On the quantum cohomology of Fano bundles over projective spaces

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Dedicated to Günther Trautmann on the occasion of his 60th birthday.

Abstract. In their paper "*Quantum cohomology of projective bundles over* $\mathbb{P}^n$" (Trans. Am. Math. Soc. (1998) 350:9 3615-3638) Z. Qin and Y. Ruan introduce interesting techniques for the computation of the quantum ring of manifolds which are projectivized bundles over projective spaces; in particular, in the case of splitting bundles they prove under some restrictions the formula of Batyrev about the quantum ring of toric manifolds. Here we prove the formula of Batyrev on the quantum Chern-Leray equation for a large class of splitting bundles.

1. Introduction

Over the last years the study of the quantum cohomology ring of projective manifolds has become a very useful tool in enumerative geometry. The theory has by now a satisfactory mathematical foundation [6, 8]. In general the actual computation is not easy, and there is more and more interest in the explicit calculation of concrete examples.

In particular Qin and Ruan in the paper [7] develop new interesting techniques for the computation of the quantum ring of manifolds which are projective bundles associated to holomorphic vector bundles on projective spaces; in the case of a splitting vector bundle the corresponding projective bundles are toric manifolds, so that the Batyrev formula [3] can be taken into account.

Qin and Ruan prove the Batyrev formula for projective splitting bundles which are Fano manifolds, under some numerical assumption on the first Chern class. Their result has been generalized in [4] to the case of splitting vector bundles over products of projective spaces.

Let $V = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ be a vector bundle of rank r on $\mathbb{P}^n$ and $X = \mathbb{P}(V)$ the corresponding projective bundle. Let h be the cohomology class of a hyperplane of $\mathbb{P}^n$ and ξ the class of the tautological line bundle on $\mathbb{P}(V)$. We make no distinction between h and $\pi^* h$ where $\pi\colon \mathbb{P}(V) \to \mathbb{P}^n$ is the projection. The cohomology ring $\mathrm{H}^*(X, \mathbb{Z})$ is generated by h and ξ with the two relations

(i) the hyperplane equation
$$h^{n+1} = 0; \tag{1}$$

(ii) the Chern-Leray equation
$$\sum_{j=0}^{r} (-1)^j c_j \xi^{r-j} h^j = \prod_{i=1}^{r} (\xi - m_i h) = 0 \tag{2}$$

($c_j = c_j(V)$ for $j = 0, \ldots, r$ are the Chern classes of V).

[1] Both authors were supported by MURST and GNSAGA-INDAM

The formula of Batyrev (as stated by Qin and Ruan) reads as follows: let $V = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ where $m_i > 0$ for every i. Then the quantum cohomology ring $Q\mathrm{H}^*(\mathbb{P}(V))$ is generated by h and ξ with two relations (to be evaluated in the quantum ring)

$$h^{n+1} = \prod_{i=1}^r (\xi - m_i h)^{m_i - 1} q_2, \tag{3}$$

$$\prod_{i=1}^r (\xi - m_i h) = q_1. \tag{4}$$

Qin and Ruan prove the above formulas under the additional assumption

$$\sum_{i=1}^r m_i < \min(2r, (n+1+2r)/2, (2n+2+r)/2). \tag{5}$$

Here we give a proof of the quantum Chern-Leray equation in the case of a bundle of the form $V = \mathcal{O}(1) \oplus \mathcal{O}(m_2) \oplus \ldots \oplus \mathcal{O}(m_r)$, where $2 \leq m_2 \ldots \leq m_r$.

Though the Batyrev formula has been proved by Givental [5] in a general context, our direct methods can be of some independent interest, because they could be applied to non-toric cases. Moreover some of our computations work in the non Fano case as well.

The more general problem of computing the quantum ring of any Fano manifold which is a projectivized bundle over a projective manifold seems at the moment out of reach; nevertheless the case of rank two bundles should be affordable, mainly because some of them have been classified: in [11] when the base space is a surface, and in [2] when the base space is a projective space or a quadric. In [1] we are able to perform the computation in the three-dimensional case.

2. Notations

We consider the projective bundle $\pi\colon \mathbb{P}(V) \to \mathbb{P}^n$ associated to a rank-r splitting vector bundle $V = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ on $\mathbb{P}^n$. We assume that the integers m_i satisfy the inequalities $1 = m_1 \leq m_2 \leq \ldots \leq m_r$. In $\mathrm{H}_2(\mathbb{P}(V), \mathbb{Z})$ we fix the classes $A_1 = (\xi^{r-2} h^n)_*$ and $A_2 = (\xi^{r-1} h^{n-1} + (1 - c_1)\xi^{r-2} h^n)_*$, where the symbol $(-)_*$ means the Poincaré dual homology class; then A_1 is the class of a line in the fiber of $\pi\colon \mathbb{P}(V) \to \mathbb{P}^n$ and A_2 is the class of any rational curve constructed as follows. Let $\ell \subset \mathbb{P}^n$ be a line; a surjective morphism $V|_\ell = \bigoplus_{i=1}^r \mathcal{O}_\ell(m_i) \to \mathcal{O}_\ell(1)$ produces an embedding

$$\tilde{\ell} = \mathbb{P}(\mathcal{O}_\ell(1)) \hookrightarrow \mathbb{P}(V|_\ell) \subset \mathbb{P}(V)$$

then $[\tilde{\ell}] = A_2$, indeed $\xi[\tilde{\ell}] = 1 = \xi \cdot A_2$ and $h[\tilde{\ell}] = h[\ell] = 1 = h[A_2]$. It is well known that A_1 and A_2 are the extremal rays of the Mori cone $\mathrm{NE}(\mathbb{P}(V))$.

The anticanonical divisor of $\mathbb{P}(V)$ is

$$-K_{\mathbb{P}(V)} = r\xi + (n + 1 - c_1(V))h$$

so that $-K_{\mathbb{P}(V)} \cdot A_1 = r$ and $-K_{\mathbb{P}(V)} \cdot A_2 = n + 1 + r - c_1$. The moduli space $\overline{M}_{0,3}(\mathbb{P}(V), B)$ of genus-0 stable maps with three marked points of class $B = aA_1 + bA_2$ has virtual dimension

$$\begin{aligned}\mathrm{virtdim}\big(\overline{M}_{0,3}(\mathbb{P}(V), B)\big) &= -K \cdot B + \dim(\mathbb{P}(V)) \\ &= ar + b(n + 1 + r - c_1) + n + r - 1.\end{aligned}$$

Let $\{T_i\}_{i=0\ldots m}$ be a $\mathbb{Z}$-basis of $\mathrm{H}^*(X, \mathbb{Z})$; in our setting we will take as $\mathbb{Z}$-basis the set of monomials $\xi^l h^m$ with $0 \leq l \leq r-1$ and $0 \leq m \leq n$. Let $\hat{T}_i$ be the cohomology class dual to T_i. The quantum product of two cohomology classes is given by

$$\alpha * \beta = \sum_{a,b} \sum_i I_{aA_1 + bA_2}(\alpha, \beta, T_i) \hat{T}_i q_1^a q_2^b,$$

where $I_{aA_1+bA_2}(\alpha,\beta,T_i)$ are the genus-0 Gromov-Witten invariant and q_1, q_2 are formal indeterminates. The degrees of q_1 and q_2 are defined as $\deg(q_1) = -K_{\mathbb{P}(V)} \cdot A_1 = r$ and $\deg(q_2) = -K_{\mathbb{P}(V)} \cdot A_2 = n+1+r-c_1$ (which allows the quantum product to be homogeneous).

Some of our computation will take place in the quotient ring $QH^*(\mathbb{P}(V))/(q_1)$, that is, we will consider in the quantum product only the quantum correction coming from the classes bA_2 with $b \geq 1$ and discard all other contributions. For $\alpha, \beta \in QH^*(\mathbb{P}(V))$ we will write $\alpha = \beta \pmod{q_1}$ to mean that α and β are equal in the quotient $QH^*(\mathbb{P}(V))/(q_1)$. The same for $\alpha = \beta \pmod{q_2}$.

3. On the quantum Chern-Leray equation

Theorem 3.1 *Let $V := \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(m_i)$. We suppose that $X = \mathbb{P}(V)$ is a Fano manifold (i.e., $c_1 \leq n+r$) and $1 = m_1 < m_2 \leq m_3 \ldots \leq m_r$. Then Batyrev formula (4)*

$$\prod_{i=1}^{r}{}^{*}(\xi - m_i h) = q_1$$

holds.

Lemma 3.2 *Let Z be the hypersurface of $\mathbb{P}(V)$*

$$Z = \mathbb{P}\big(\textstyle\bigoplus_{i=2}^r \mathcal{O}(m_i)\big) \hookrightarrow \mathbb{P}(V)$$

corresponding to the canonical projection $V \to \bigoplus_{i=2}^r \mathcal{O}(m_i)$. Then for every curve T of class bA_2 with $b \geq 1$ we have $T \cap Z = \emptyset$.

Proof: By induction on b, we can suppose T irreducible. Let us suppose by contradiction $T \cap Z \neq \emptyset$. Since $[Z] = \xi - h$ and $(\xi - h)\cdot T = 0$ we conclude that $T \subset Z$. Let $\xi_Z = \mathcal{O}_{\mathbb{P}(F(-m_2+1))}(1)$ be the tautological line bundle on $\mathbb{P}(F(-m_2+1))$ where $F(-m_2+1) = \mathcal{O}(1) \oplus \mathcal{O}(1-m_2+m_3) \oplus \ldots \oplus \mathcal{O}(1-m_2+m_r)$. Hence

$$\xi|_Z = \xi_Z + (m_2 - 1)h;$$

then $0 = (\xi - h)\cdot T = (\xi_Z + (m_2-2)h)\cdot T$ so that $\xi_Z \cdot T = b(2-m_2) \leq 0$ which contradicts the ampleness of ξ_Z. Q.E.D.

Lemma 3.3 *For $s \leq r-1$ and $l \leq n$ the class $(\xi - h)\xi^s h^l$ is represented by a subvariety of X contained in Z.*

Proof: Let $p = \pi|_Z : Z \to \mathbb{P}^n$ be the projective subbundle of $\mathbb{P}(V)$. The class $h^l(\xi - h)$ is represented by $p^{-1}(\mathbb{P}_0^{n-l})$, where $\mathbb{P}_0^{n-l}$ is a linear subspace of $\mathbb{P}^n$, so that the class $(\xi-h)\xi^s h^l$ coincides with the class $\xi^s p^{-1}(\mathbb{P}_0^{n-l}) = \xi^s|_Z \cdot p^{-1}(\mathbb{P}_0^{n-l})$ which is represented by an effective subvariety of Z, because $\xi|_Z$ is ample and generated by global sections. Q.E.D.

Corollary 3.4 *For any integer $b \geq 1$ and any triple $\gamma_1, \gamma_2, \gamma_3$ of cohomology classes in X the following equality holds:*

$$I_{bA_2}\big((\xi - h)\gamma_1, \gamma_2, \gamma_3\big) = 0.$$

Proof: By the linearity of the Gromov-Witten invariants we can suppose $\gamma_1 = \xi^s h^l$, with $s \leq r-1$ and $l \leq n$. By 3.2 and 3.3 the class $(\xi - h)\gamma_1$ is represented by a subvariety which intersects no curves of class bA_2. Q.E.D.

Lemma 3.5 *For $p + q = m + t$ and $p \leq r$, $m \leq r$, the following equality holds:*

$$\xi^{*p} * h^{*q} - \xi^{*m} * h^{*t} = \xi^p h^q - \xi^m h^t \pmod{q_1}.$$

Proof: It is enough to show

$$\xi^{*(l+1)} * h^{*s} - \xi^{*l} * h^{*(s+1)} = \xi^{l+1}h^s - \xi^l h^{s+1} \pmod{q_1}$$

for any $l, s \geq 0, l \leq r-1$. First we suppose $s > 0$; let $\gamma_1 := \xi^{*l} * h^{*s}$, $\gamma_2 := \xi^{*(l+1)} * h^{*(s-1)}$. By induction on $l+s$ we obtain

$$\gamma_1 - \gamma_2 = \xi^l h^s - \xi^{l+1} h^{s-1} \pmod{q_1}.$$

Then

$$\begin{aligned}
\xi^{*(l+1)} * h^{*s} - \xi^{*l} * h^{*(s+1)} &= (\gamma_1 - \gamma_2) * h \pmod{q_1} \\
&= \xi^{l+1} h^s - \xi^l h^{s+1} + \sum_{b \geq 1} \sum_i I_{bA_2}(\xi^l h^s - \xi^{l+1} h^{s-1}, h, T_i) \hat{T}_i q_2^b \pmod{q_1}.
\end{aligned}$$

By corollary 3.4, all the $I_{bA_2}(\xi^l h^s - \xi^{l+1} h^{s-1}, h, T_i) = I_{bA_2}(-(\xi - h)\xi^l h^{s-1}, h, T_i)$ vanish. The same computation, interchanging the role of ξ and h works if $l > 0$. The conclusion follows. Q.E.D.

Proof of theorem 3.1: Since $\deg(q_1) = r$, if $0 \leq i \leq r$, it follows from lemma 3.5 that

$$\xi^{*(r-i)} * h^{*i} - h^{*r} = \xi^{r-i} h^i - h^r + t_i q_1$$

for some integers t_i. By lemma 3.5 and 3.7 of [7], $I_{A_1}(\xi, \xi^{r-1}, \xi^{r-1} h^n) = 1$ and $I_{A_1}(\xi^{l_1} h^{m_1}, \xi^{l_2} h^{m_2}, \alpha) = 0$ for $(l_1 + l_2) < r$ and any $\alpha \in \mathrm{H}^*(\mathbb{P}(V))$. So

$$\xi^{*(r-i)} * h^{*i} = \xi^{r-i} h^i + \begin{cases} q_1 \pmod{q_2} & \text{if } i = 0 \\ 0 \pmod{q_2} & \text{if } 1 \leq i \leq r \end{cases}$$

that is, $t_0 = 1$ and $t_i = 0$ for $1 \leq i \leq r$. Let $\bar{c}_V(t)$ be the polynomial

$$\bar{c}_V(t) = \sum_{j=0}^{r} \bar{c}_j t^j = \prod_{i=1}^{r} (1 - m_i t).$$

Then $\bar{c}_j$ coincides with the Chern class c_j of V for $0 \leq j \leq \min(n, r)$. Since $m_1 = 1$, $\bar{c}_V(1) = 0$, that is,

$$\sum_{i=0}^{r} (-1)^i \bar{c}_i = 0.$$

It follows

$$\begin{aligned}
\prod_{i=1}^{r}{}^{*} (\xi - m_i h) &= \sum_{i=0}^{r} (-1)^i \bar{c}_i \xi^{*(r-i)} * h^{*i} \\
&= \sum_{i=0}^{r} \Big[(-1)^i \bar{c}_i \cdot (\xi^{r-i} h^i + h^{*r} - h^r + t_i q_1) \Big] \\
&= \Big(\sum_{i=0}^{r} (-1)^i \bar{c}_i \Big) \cdot (h^{*r} - h^r) + q_1 \\
&= q_1.
\end{aligned}$$

Q.E.D.

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